Fieldwork Surveying FS01

3. Lecture

Evaluation of precision and accuracy of a measurement

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Accuracy × precision

The term **accuracy** refers to the closeness between measurements and their true values. The further a measurement is from its true value, the less accurate it is.

As opposed to accuracy, the term **precision** is related to the closeness to one another of a set of repeated observation of a random variable. If such observations are closely gathered together, then they are said to have been obtained with high precision.

It should be apparent that observations may be precise but not accurate, if they are closely grouped together but about a value that is different from the true value by a significant amount.

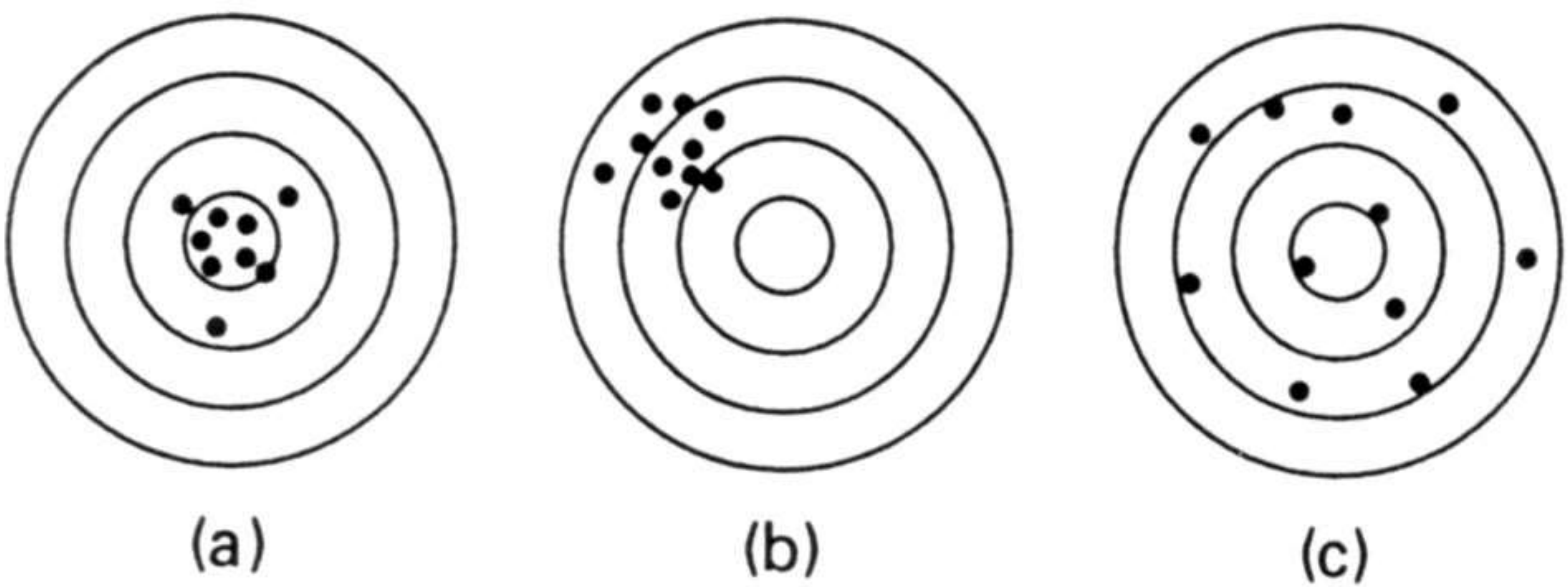
Also, observations may be accurate but not precise if they are well distributed about the true value but dispersed significantly from one another.

Example – rifle shots

(a) – both accurate and precise

(b) – precise but not accurate

(c) – accurate but not precise



Another definition - metrological

- Accuracy of measurement

closeness of agreement between a **measured quantity value** and a **true quantity value** of a **measurand**

- Trueness of measurement

closeness of agreement between the average of an infinite number of replicate **measured quantity values** and a **reference quantity value**

- Precision

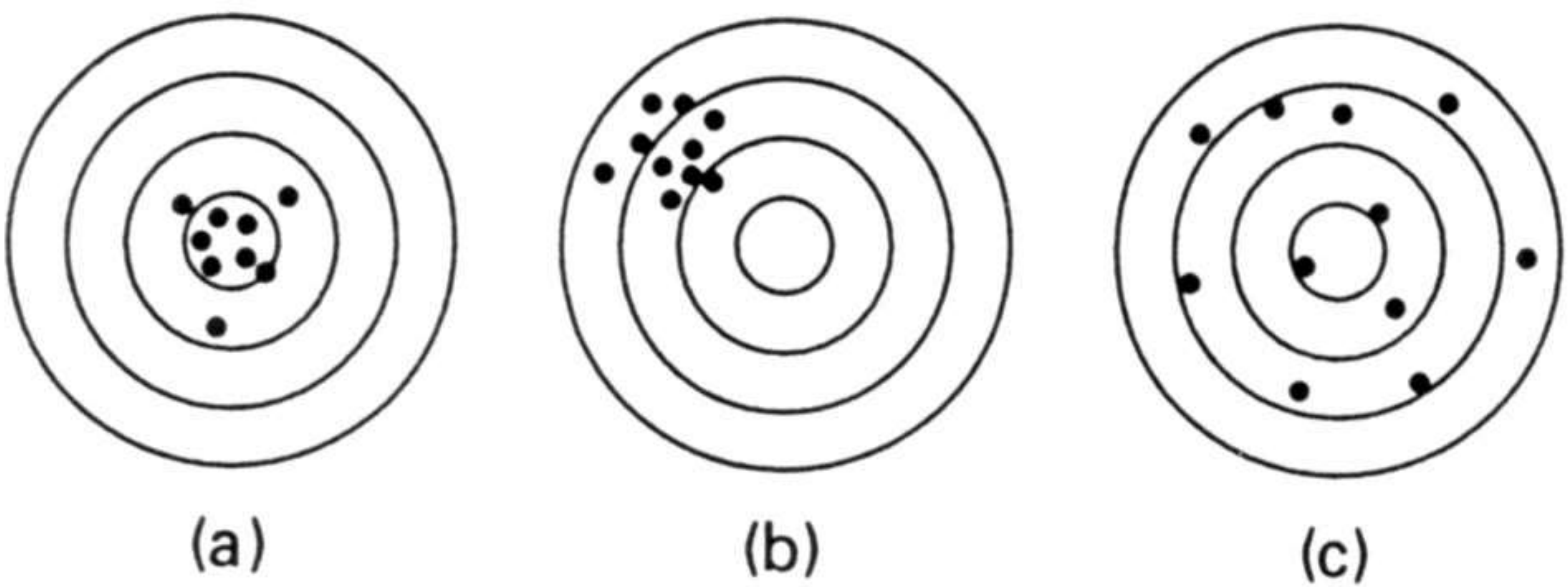
closeness of agreement between **indications** or **measured quantity values** obtained by replicate **measurements** on the same or similar objects under specified conditions

Example – rifle shots

(a) both trueness and precise

(b) precise but not trueness

(c) trueness but not precise



Errors

Errors are considered to be of three types:

1. mistakes,
2. gross errors,
3. systematic errors,
4. random errors.

Mistakes

Mistakes actually are not errors because they usually are so gross in size compared to the other two types of errors. One of the most common reasons for mistakes is simple carelessness of the observer, who may take the wrong reading of a scale or, if the proper value was read, may record the wrong one by transposing numbers, for example.

Gross errors

Gross errors are errors which they usually are so gross in size compared to the other two types of errors. One of the most common reasons for gross errors is cumulating of disadvantageous influences or their abnormal size (strong wind, atmospheric refraction, vibrations.

We don’t calculate with mistakes and gross errors.

$$\varepsilon_i = \Delta_i + c_i$$

ε_i … real error of a measurement

Δ_i … random error

c_i … systematic error

$$\varepsilon_i = X - l_i$$

X … true (real) value of a quantity

l_i … measured value of a quantity

(reference quantify value minus a measured quantify value is measurement error)

Systematic errors

Systematic errors or effects occur according to a system that, if known, always can be expressed by a mathematical formulation. Any change in one or more of the elements of the system will cause a change in the character of the systematic effects if the observational process is repeated. Systematic errors occur due to natural causes, instrumental factors and the observer's human limitation.

Random errors

Random errors are unavoidable.

It is component of measurement error that in replicate measurements varies in an unpredictable manner.

The precision depends only on parameters of random errors distribution.

Example:

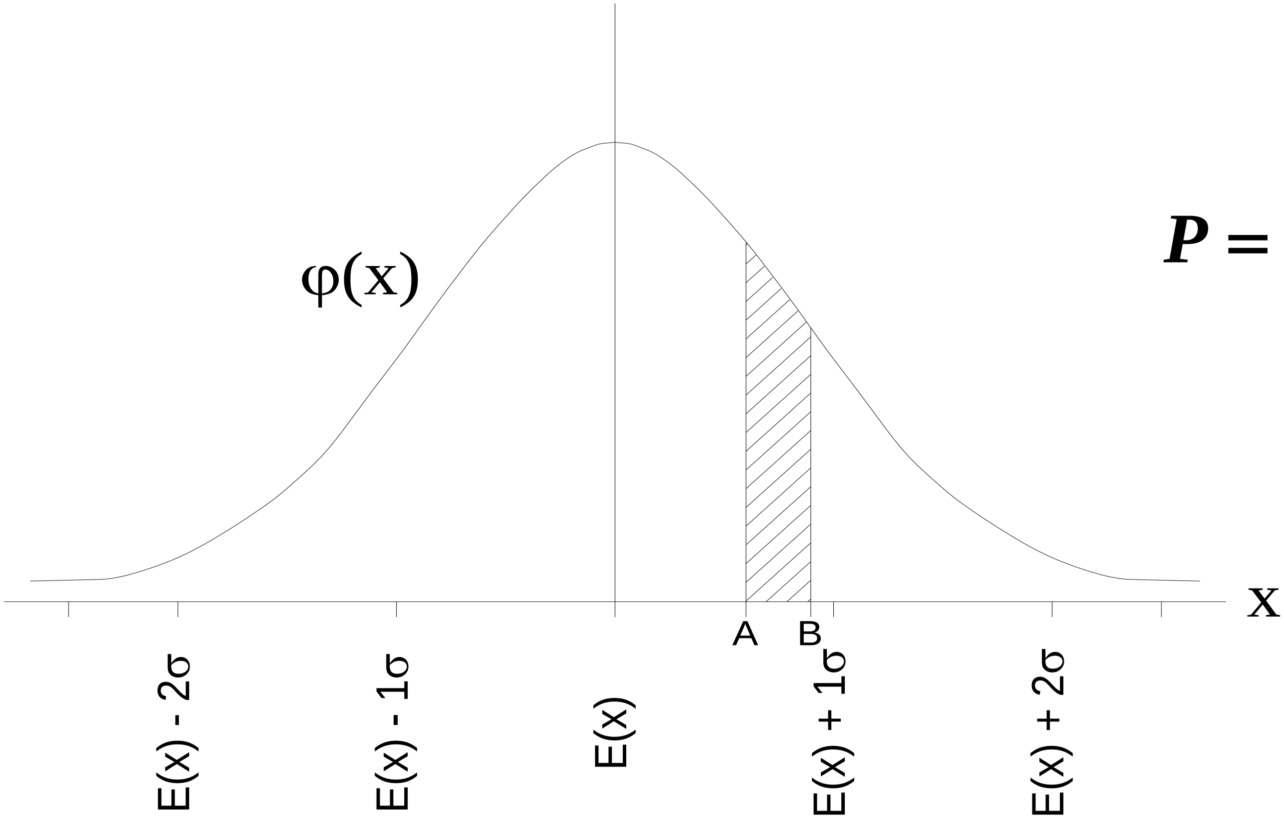
If the reference value of an angle is 41,1336 gon and the angle is measured 20 times, it is usual to get values for each of the measurements that differ slightly from the true angle. Each of these values has a probability that it will occur. The closer the value approaches reference value, the higher is the probability, and the farther away it is, the lower is the probability.

Characteristics of random errors:

- probability of a plus and minus error of the certain size is the same,
- probability of a small error is higher than probability of a large error,
- errors larger than a certain limit do not occur (they are considered to be mistakes).

The distribution of random errors is called normal (Gauss) distribution and probability density of normal distribution is interpreted mathematically with the frequency curve of the normal distribution (Gauss curve) $\rightarrow$

Frequency curve of the normal distribution



$$P = \int_{A}^{B} \varphi(x)\,dx$$

The probability P that the measurement will be affected by an error of size falling in the interval <A;B> is equal to the area hatched in the graph. The probability can be in the interval <0;1>.

Several values of the probabilities P characterizing the normal distribution:

A	B	P
E(x)	E(x) + σ	0,341
E(x) - σ	E(x) + σ	**0,682**
E(x)	E(x) + 2σ	0,477
E(x) -2σ	E(x) + 2σ	**0,954**
E(x)	E(x) + 3σ	0,499
E(x) – 3σ	E(x) + 3σ	**0,997**
E(x) – ∞	E(x) + ∞	1,000

E(x) is mean value (unknown true value of a quantity), It is characteristic of location.

$\boldsymbol{\sigma^2}$ is variance = square of standard deviation. It is characteristic of variability

It is not possible to find out the true value of a quantity. The result of a measurement is the most reliable value of a quantity and its accuracy.

Processing of measurements with the same accuracy

Characteristic of measurement precision - standard deviation.

$$\sigma = \sqrt{\frac{[\varepsilon\varepsilon]}{n}} = \sqrt{\frac{\sum_{i=1}^{n} \varepsilon_i^{\,2}}{n}}$$

n … number of measurements,

σ … if $n \to \infty$ (standard deviation),

s … if *n* is smaller number (sample standard deviation).

The true value of a quantity *X* is unknown (→ the real error *ε* is also unknown). Therefore the **arithmetic mean** is used as the most probable estimation of *X*. The difference between the average and a particular measurement is called correction v_i. The sample standard deviation *s* is calculated using these corrections.

$$\bar{l} = \frac{[l]}{n} = \frac{\sum_{i=1}^{n} l_i}{n}$$

$$v_i = \bar{l} - l_i$$

$$s = \sqrt{\frac{[vv]}{n-1}} = \sqrt{\frac{\sum_{i=1}^{n} {v_i}^2}{n-1}}$$

If the standard deviation of one measurement is known and the measurement is carried out more than once, the standard deviation of the average is calculated according to the formula

$$\boldsymbol{\sigma_{\bar{l}} = \frac{\sigma}{\sqrt{n}}}$$

$$s_{\bar{l}} = \frac{s}{\sqrt{n}}$$

Processing of measurements with the same accuracy – example

Problem: a distance was measured 5x under the same conditions and by the same method (= with the same accuracy).

Measured values are: 5,628; 5,626; 5,627; 5,624; 5,628 m.

Calculate the average distance, the standard deviation of one measurement and the standard deviation of the average.

Solution:

i	l / m	v / m	vv / m^2
1	**5,628**	-0,0014	1,96E-06
2	**5,626**	0,0006	3,60E-07
3	**5,627**	-0,0004	1,60E-07
4	**5,624**	0,0026	6,76E-06
5	**5,628**	-0,0014	1,96E-06
Σ	28,133	0,000	1,12E-05
$\bar{l}$ = 5,6266 m, s_{l_i} = 0,0017 m, $s_{\bar{l}}$ = 0,00075 m			

Law of standard deviation propagation

Sometimes it is not possible to measure a value of a quantity directly and then this value is determined implicitly by calculation using other measured values.

Examples:

- area of a triangle – two distances and an angle are measured,
- height difference – a slope distance and a zenith angle are measured.

If a mathematical relation between measured and calculated values is known, the standard deviation of the calculated value can be derived using the law of standard deviation propagation.

If a mathematical relation is known

$$y = f(x_1, x_2, x_3, x_4, ..., x_k)$$

then

$$y + \varepsilon_y = f(x_1 + \varepsilon_{x_1}, x_2 + \varepsilon_{x_2}, x_3 + \varepsilon_{x_3}, x_4 + \varepsilon_{x_4}, ..., x_k + \varepsilon_{x_k})$$

Real errors are much smaller than measured values therefore the right part of the equation can be expanded according to the Taylor's expansion (using terms of the first degree only).

$$y + \varepsilon_y = f(x_1, x_2, x_3, ..., x_k) + \frac{\partial f}{\partial x_1}\varepsilon_{x_1} + \frac{\partial f}{\partial x_2}\varepsilon_{x_2} + \frac{\partial f}{\partial x_3}\varepsilon_{x_3} + ... + \frac{\partial f}{\partial x_k}\varepsilon_{x_k}$$

Law of real error propagation:

$$\varepsilon_y = \frac{\partial f}{\partial x_1}\varepsilon_{x_1} + \frac{\partial f}{\partial x_2}\varepsilon_{x_2} + \frac{\partial f}{\partial x_3}\varepsilon_{x_3} + ... + \frac{\partial f}{\partial x_k}\varepsilon_{x_k}$$

Real errors of measured values are not usually known but standard deviations are known.

The law of standard deviation propagation:

$$\sigma_y^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x_2}^2 + \left(\frac{\partial f}{\partial x_3}\right)^2 \sigma_{x_3}^2 + \dots + \left(\frac{\partial f}{\partial x_k}\right)^2 \sigma_{x_k}^2$$

It is possible to use the law of standard deviation propagation if only:

1. Measured quantities are mutually independent.
2. Real errors are random.
3. Errors are much smaller than measured values.
4. There is the same unit of all terms.

Examples

Problem 1: the formula for a calculation of the standard deviation of the average should be derived if the standard deviation of one measurement is known and all measurements were carried out with the same accuracy

Solution:

$$\bar{l} = \frac{l_1 + l_2 + ...l_n}{n}$$

$$\varepsilon_{\bar{l}} = \frac{1}{n}\left\{\varepsilon_{l_1} + \varepsilon_{l_2} + ... + \varepsilon_{l_n}\right\}$$

$$\sigma_{\bar{l}}^2 = \frac{1}{n^2}\left\{\sigma_{l_1}^2 + \sigma_{l_2}^2 + ... + \sigma_{l_n}^2\right\}$$

The real errors are random and generally different

$$\varepsilon_{l_1} \neq \varepsilon_{l_2} \neq \ldots \neq \varepsilon_{l_n}$$

The standard deviations are the same for all measurements

$$\sigma_{l_1}^2 = \sigma_{l_2}^2 = \ldots = \sigma_{l_n}^2 = \sigma_l^2$$

therefore

$$\sigma_{\bar{l}}^2 = \frac{1}{n^2}\left\{\sigma_l^2 + \sigma_l^2 + \ldots + \sigma_l^2\right\} = \frac{1}{n^2}\left\{n \cdot \sigma_l^2\right\} = \frac{n}{n^2}\left\{\sigma_l^2\right\} = \frac{1}{n}\sigma_l^2$$

$$\sigma_{\bar{l}} = \frac{1}{\sqrt{n}}\sigma_l$$

Problem 2: two distances were measured in a triangle a = 34,352 m, b = 28,311 m with the accuracy $\sigma_a = \sigma_b = 0,002$ m. The horizontal angle was also measured $\omega = 52,3452°$, $\sigma_\omega = 0,0045°$. Calculate the standard deviation of the area of the triangle.

Solution:

$$P = \frac{1}{2} a \cdot b \cdot sin(\omega)$$

$$\varepsilon_P = \frac{1}{2} b \cdot sin(\omega) \cdot \varepsilon_a + \frac{1}{2} a \cdot sin(\omega) \cdot \varepsilon_b + \frac{1}{2} a \cdot b \cdot cos(\omega) \cdot \varepsilon_\omega \cdot \frac{\pi}{180°}$$

$$\left(\rho° = \frac{180°}{\pi} \right)$$

$$\sigma_P^2 = \frac{1}{4}\left(b \cdot sin(\omega)\right)^2 \cdot \sigma_a^2 + \frac{1}{4}\left(a \cdot sin(\omega)\right)^2 \cdot \sigma_b^2 + \frac{1}{4}\left(a \cdot b \cdot cos(\omega)\right)^2 \cdot \frac{\sigma_\omega^2}{\left(\rho^\circ\right)^2}$$

$\sigma_a = \sigma_b = \sigma_d$:

$$\sigma_P^2 = \frac{1}{4}\left(b^2 + a^2\right) \cdot sin^2(\omega) \cdot \sigma_d^2 + \frac{1}{4}\left(a \cdot b \cdot cos(\omega)\right)^2 \cdot \frac{\sigma_\omega^2}{\left(\rho^\circ\right)^2}$$

$\sigma_P = 0{,}043\ m^2$.